ALGEBRA PRELIMINARY EXAM
AUGUST 22, 2026

All responses require justification.

(1) Let G be an abelian group, and let H be a subgroup of G. Let $S = \{a \in G : a^2 \in H\}$. Prove that S is a subgroup of G.

(2) Let R denote an integral domain. Let J be a nontrivial ideal of R that is also a field. Prove that $J = R$.

(3) Let $A = \begin{pmatrix} -2 & 3 & 0 \\ -6 & 7 & 0 \\ 0 & 0 & 16 \end{pmatrix}$. Find a matrix B such that $B^4 = A$.

(4) Suppose that A is an $n \times k$ real matrix such that the span of the column vectors of A is $\mathbb{R}^n$. Let $L_A : \mathbb{R}^n \to \mathbb{R}^k$ be the transformation defined by $L_A(x) = A^T x$. Prove that L_A is 1-1.

(5) Let $g(x) = x^3 + 3x - 3$.

(a) Find the Galois group of its splitting field over $\mathbb{Q}$.

(b) Find the Galois group of its splitting field over $\mathbb{F}_7$.

(6) For any ring R, let the subset $R_0 \subseteq R$ be defined by
$R_0 = \{x \in R : x^k = 0 \text{ for some } k \in \{1, 2, 3, \ldots\}\}$.

(a) Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} : a, b, c, d, e, f \in \mathbb{R} \right\}$, with the usual matrix addition and multiplication being the ring operations. Find R_0 for this ring.

(b) Prove that R_0 in (a) above is a two-sided ideal in R.

(7) Show that $\mathbb{Q}(\sqrt{2} + i) = \mathbb{Q}(\sqrt{2}, i)$, and find the minimal polynomial of $\sqrt{2} + i$ over $\mathbb{Q}$.

(8) Let F be a field, and let a be an element in an extension field of F. Suppose that $[F(a^4) : F]$ is finite, and let $b \in F(a) \setminus F$. Prove that b is algebraic over F.